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LETTER TO THE EDITOR

Equivalence between tunnelling times based on (a) absorption probabilities, (b) the Larmor clock, and (c) scattering projectors

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Abstract. The assumption of analyticity of the transmission and reflection probability amplitudes as functions of a complex potential is shown to be justified. As a consequence the reflection and transmission times based on absorption probabilities are shown to be equal to the corresponding times derived from the local Larmor precession in the plane perpendicular to the magnetic field. An additional new interpretation of these times is provided by means of a more general scattering theory projector formalism. The relation to the phase times is discussed.

Considerable attention is being paid to the theoretical and experimental determination of quantum interaction times, especially in the context of one-dimensional scattering, because of the possible technological implications in resonant tunnelling devices. While there is general agreement on the meaning of the dwell time [1], which for the stationary scattering wave ψ takes the form

$$\tau_D(x_1, x_2) = F_\mathrm{I}^{-1}\langle\psi|D(x_1, x_2)|\psi\rangle \tag{1}$$

where F_I is the incident flux, and $D(x_1, x_2)$ is the projector selecting the coordinate space interval $[x_1, x_2]$

$$D \equiv D(x_1, x_2) \equiv \int_{x_1}^{x_2} \mathrm{d}x\, |x\rangle\langle x| \tag{2}$$

the dwell time decomposition into transmission and reflection times is controversial, and a diversity of approaches claim to hold the right or best answer (see the recent reviews [2–4]). Clearly it is important to discern the differences and similarities between them. Our effort here is in this direction.

Büttiker [3, 5], and Huang and Wang [6] have shown that the partial times based on absorption probabilities, $\tau_\mathrm{T}^\mathrm{a}$ and $\tau_\mathrm{R}^\mathrm{a}$, are equal to the local Larmor times, $\tau_{x\mathrm{T}}$ and $\tau_{x\mathrm{R}}$ [1, 3, 7]. (The definition of the local Larmor times requires a gedanken clock experiment consisting in an incident beam fully spin-polarized in the y direction impinging on the barrier from the left in the x direction in the presence of an infinitesimal uniform magnetic field $B = B\Theta(x - x_1)\Theta(x_2 - x)\hat{z}$ covering the interval $[x_1, x_2]$.) However, their result relies on the assumption that the transmission and reflection probability amplitudes are complex analytic functions of the complex potential as made explicit in [6]. In this letter it is first shown that this assumption is entirely justified, thus confirming the results of Büttiker, and Huang and Wang. In addition, we provide a scattering projector formalism, sketched in [8], that includes *all* Larmor times (and consequently the absorption-based times) as particular cases. The relation to phase delay times is also indicated. We restrict ourselves to stationary scattering.

The first step is the evaluation of the transmission and reflection coefficients in terms of the transition matrix. Assume that a complex potential ΔV acts between x_1 and x_2, with $\Delta V_{\mathrm{Re}} \equiv \mathrm{Re}(\Delta V)$ and $\Delta V_{\mathrm{Im}} \equiv \mathrm{Im}(\Delta V)$ as real positive numbers independent of x. The full potential is the sum of $\mathcal{U}(x) = \Delta V\Theta(x - x_1)\Theta(x_2 - x)$, and the physical interaction $V_0(x)$

$$V(x) = V_0(x) + \mathcal{U}(x). \tag{3}$$

By *physical* we mean that $V_0(x)$ is the actual potential of our system. The complex potential is here regarded as an auxiliary theoretical entity. Eventually we will be interested in the limit where this complex potential vanishes.

The Lippmann–Schwinger equation for $V(x)$ is not formally affected by the presence of the imaginary term. Thus the wave function

$$\langle x \mid p^{(+)}\rangle = \langle x \mid p\rangle + \lim_{\epsilon\to 0}\langle x|G_0(E_p + \mathrm{i}\epsilon)T(E_p + \mathrm{i}\epsilon)|p\rangle \tag{4}$$

(ϵ and p will always be taken as positive) is a solution of the Schrödinger equation having energy $E_p = p^2/(2m)$ and outgoing boundary conditions for the scattered wave, with incident plane wave $\langle x \mid p\rangle = h^{-1/2}\exp(\mathrm{i}px/\hbar)$ [9]; $T(z) = V\Omega(z)$ is the energy-parametrized transition operator; $\Omega(z) = 1 + G_0(z)T(z)$ is a parametrized Möller operator (see [10] for a full account on different parametrizations), and $G_0(z) = (z - H_0)^{-1}$ is the resolvent of the kinetic energy operator H_0.

We are particularly interested in the behaviour at asymptotic positions

$$\langle x \mid p^{(+)}\rangle_{x\to-\infty} \sim h^{-1/2}[\exp(\mathrm{i}px/\hbar) + R_p\exp(-\mathrm{i}px/\hbar)]$$
$$\langle x \mid p^{(+)}\rangle_{x\to\infty} \sim h^{-1/2}T_p\exp(\mathrm{i}px/\hbar) \tag{5}$$

where T_p and R_p are the complex transmission and reflection coefficients (probability amplitudes) respectively. Explicit expressions for them in terms of matrix elements of the transition operator are found by considering, using contour integration in the complex momentum plane, the asymptotic behaviour of the Green's function in (4), and comparing with (5),

$$T_p = 1 - (2m\mathrm{i}\pi/p)\langle p|T|p\rangle \tag{6}$$

$$R_p = -(2m\mathrm{i}\pi/p)\langle -p|T|p\rangle. \tag{7}$$

See [11] for an alternative derivation. The coefficients T_p and R_p depend on the matrix elements of T on the energy shell; in other words, the energy of the ket and bra momenta is equal to the real part of the argument of $T(z)$. (Do not miss the transition operator T, the transmission coefficient T_p and the transition matrix $T_{pp'} = \lim_{\epsilon\to 0}\langle p|T(E_{p'} + \mathrm{i}\epsilon)|p'\rangle$.)

Suppose that V depends on a parameter α. Then the derivative of the on-shell matrix elements of T with respect to α is given by [12]

$$\mathrm{d}T_{p'p}/\mathrm{d}\alpha = \langle p'^{(-)}|\mathrm{d}V/\mathrm{d}\alpha|p^{(+)}\rangle \tag{8}$$

where $|p^{(-)}\rangle = \lim_{\epsilon\to 0}\Omega(E_p - \mathrm{i}\epsilon)|p\rangle$. Now take $\alpha = \Delta V$. The analyticity of $T_{p'p}$, and therefore of R_p and T_p, with respect to the parameter ΔV is thus determined by the analyticity of V with respect to ΔV. But the derivative $\mathrm{d}V/\mathrm{d}(\Delta V)$ exists for all ΔV, and equals the projector operator selecting the coordinate space interval $[x_1, x_2]$,

$$\mathrm{d}V/\mathrm{d}(\Delta V) = D(x_1, x_2). \tag{9}$$

This implies the validity of the Cauchy–Riemman conditions required in [6] to equate the local Larmor clock times, $\tau_{x\mathrm{T}}$ and $\tau_{x\mathrm{R}}$, and the absorption based times, $\tau^{\mathrm{a}}_{\mathrm{T}}$ and $\tau^{\mathrm{a}}_{\mathrm{R}}$.

In particular,

$$\begin{aligned}
\left.\frac{\partial\,\mathrm{Re}\,T_{pp'}}{\partial\Delta V_{\mathrm{Re}}}\right|_{\Delta V=0} &= \left.\frac{\partial\,\mathrm{Im}\,T_{pp'}}{\partial\Delta V_{\mathrm{Im}}}\right|_{\Delta V=0} = \mathrm{Re}\,\langle p^{(-)}|D|p'^{(+)}\rangle|_{\Delta V=0} \\
\left.\frac{\partial\,\mathrm{Re}\,T_{pp'}}{\partial\Delta V_{\mathrm{Im}}}\right|_{\Delta V=0} &= -\left.\frac{\partial\,\mathrm{Im}\,T_{pp'}}{\partial\Delta V_{\mathrm{Re}}}\right|_{\Delta V=0} = -\mathrm{Im}\,\langle p^{(-)}|D|p'^{(+)}\rangle|_{\Delta V=0}
\end{aligned} \tag{10}$$

and, using (6), one obtains the relations

$$\begin{aligned}
\left.\frac{\partial\,\mathrm{Re}\,T_{p}}{\partial\Delta V_{\mathrm{Re}}}\right|_{\Delta V=0} &= \left.\frac{\partial\,\mathrm{Im}\,T_{p}}{\partial\Delta V_{\mathrm{Im}}}\right|_{\Delta V=0} = \frac{2m\pi}{p}\,\mathrm{Im}\,\langle p^{(-)}|D|p^{(+)}\rangle|_{\Delta V=0} \\
\left.\frac{\partial\,\mathrm{Im}\,T_{p}}{\partial\Delta V_{\mathrm{Re}}}\right|_{\Delta V=0} &= -\left.\frac{\partial\,\mathrm{Re}\,T_{p}}{\partial\Delta V_{\mathrm{Im}}}\right|_{\Delta V=0} = -\frac{2m\pi}{p}\,\mathrm{Re}\,\langle p^{(-)}|D|p^{(+)}\rangle|_{\Delta V=0}.
\end{aligned} \tag{11}$$

When substituting them into the local Larmor transmission time associated with spin rotation in the $x-y$ plane [7]

$$\tau_{xT} = -\hbar\left(\frac{\partial\phi_{\mathrm{T}}(p)}{\partial\Delta V_{\mathrm{Re}}}\right)_{\Delta V=0} = \frac{\hbar}{|T_p|^2}\left[(-\mathrm{Re}\,T_p)\frac{\partial\,\mathrm{Im}\,T_p}{\partial\Delta V_{\mathrm{Re}}} + (\mathrm{Im}\,T_p)\frac{\partial\,\mathrm{Re}\,T_p}{\partial\Delta V_{\mathrm{Re}}}\right]_{\Delta V=0} \tag{12}$$

where $T_p = |T_p|\exp[\phi_{\mathrm{T}}(p)]$, the absorption based transmission time [6],

$$\tau^{\mathrm{a}}_{\mathrm{T}} = (\hbar/|T_p|^2)\left[(\mathrm{Re}\,T_p)\partial\,\mathrm{Re}\,T_p/\partial\Delta V_{\mathrm{Im}} + (\mathrm{Im}\,T_p)\partial\,\mathrm{Im}\,T_p/\partial\Delta V_{\mathrm{Im}}\right]_{\Delta V=0} \tag{13}$$

is recovered. Following a similar calculation it is also found that $\tau^{\mathrm{a}}_{\mathrm{R}} = \tau_{x\mathrm{R}}$.

The previous Larmor transmission time, equation (12), and the one associated with spin precession in the $y-z$ plane,

$$\tau_{z\mathrm{T}} = -\hbar\left(\frac{\partial\log|T_p|}{\partial\Delta V_{\mathrm{Re}}}\right)_{\Delta V=0} = \frac{\hbar}{|T_p|^2}\left[(-\mathrm{Re}\,T_p)\frac{\partial\,\mathrm{Re}\,T_p}{\partial\Delta V_{\mathrm{Re}}} + (\mathrm{Im}\,T_p)\frac{\partial\,\mathrm{Im}\,T_p}{\partial\Delta V_{\mathrm{Re}}}\right]_{\Delta V=0} \tag{14}$$

will now be obtained by means of a completely different approach based on resolving the operator D in (1) with the aid of the complementary scattering projector operators P and $Q = 1 - P$,

$$P = \int_0^\infty \mathrm{d}p\, |p^{(-)}\rangle\langle p^{(-)}|. \tag{15}$$

The projector P selects the part of a square integrable wave ϕ that will be transmitted (will have positive momentum) in the future, $P\phi$. Since P and Q commute with the Hamiltonian, the components $P\phi$ and $Q\phi$ evolve independently of each other. Three useful resolutions are:

$$D = PDP + PDQ + QDP + QDQ \tag{16}$$

$$D = PD + QD \tag{17}$$

$$D = DPD + DQD. \tag{18}$$

By inserting these expressions into (1) different decompositions of the dwell times are obtained. Equation (17) will provide complex times since PD and QD are not Hermitian. In fact this decomposition gives the complex times associated with the Larmor clock.

Let us note before that, even though, as stated, the projectors P, Q originate in wave packet scattering, they have a physical interpretation in the stationary case. Similarly, the abstract Möller operator $\Omega^{\pm} = \lim_{t\to\mp\infty} \exp(\mathrm{i}Ht)\exp(-\mathrm{i}H_0 t)$ is only meaningful for wave packets, but, when properly parametrized (e.g. in $\lim_{\epsilon\to 0}\Omega(E_p \pm \mathrm{i}\epsilon)|p\rangle$), it may act on plane waves giving continuum eigenstates of the full Hamiltonian H. These are not square integrable states, but are widely recognized as useful entities, not only mathematically, forming a basis, but because of the physical content. The states $|p^{(+)}\rangle$ describe the collision due to a constant flux F_I of incident monoenergetic particles giving a scattered wave with outgoing boundary conditions. The states $|p^{(-)}\rangle$ imply the opposite order of events, having ingoing boundary conditions for the scattered wave, and an outgoing plane wave with flux F_O equal to F_I. Let us examine the effect of P on $|p^{(+)}\rangle$. Using the definition (15), the relation

$$S_{p'p} \equiv \langle p'^{(-)}|p^{(+)}\rangle = \delta(p - p') - 2\mathrm{i}\pi\delta(E_p - E_{p'})T_{p'p} \tag{19}$$

between the S- and T-matrix elements, and (6), we obtain $P|p^{(+)}\rangle = T_p|p^{(-)}\rangle$. One may then normalize these projections in the same way as the states $|p^{(\pm)}\rangle$, $|Pp_\mathrm{N}^{(+)}\rangle \equiv |T_p|^{-1}P|p^{(+)}\rangle$. With this normalization, the decomposition of the dwell time based on (16) can be written as

$$\tau_\mathrm{D} = |T_p|^2\tau_\mathrm{T}^{PDP} + |R_p|^2\tau_\mathrm{R}^{QDQ} + \tau_\mathrm{int} \tag{20}$$

where $\tau_\mathrm{T}^{PDP} \equiv F_\mathrm{I}^{-1}\langle Pp_\mathrm{N}^{(+)}|D|Pp_\mathrm{N}^{(+)}\rangle$ and $\tau_\mathrm{R}^{QDQ} \equiv F_\mathrm{I}^{-1}\langle Qp_\mathrm{N}^{(+}|D|Qp_\mathrm{N}^{(+)}\rangle$, with $|Qp_\mathrm{N}^{(+)}\rangle \equiv |R_p|^{-1}Q|p^{(+)}\rangle$. The interference term $\tau_\mathrm{int} \equiv 2F_\mathrm{I}^{-1}\,\mathrm{Re}\,\langle p^{(+)}|PDQ|p^{(+)}\rangle$ is a quantum mechanical feature without classical counterpart. It is interesting to note that the PDP and QDQ terms are real, positive, and interpretable as transmission and reflection *dwell times*.

Using the resolution (17), instead of (16), to decompose the dwell time, complex times can be defined as

$$\tau_{\rm T}^{PD} \equiv \langle p^{(+)}|PD|p^{(+)}\rangle / F_{\rm I}|T_p|^2 \qquad \tau_{\rm R}^{PD} \equiv \langle p^{(+)}|QD|p^{(+)}\rangle / F_{\rm I}|R_p|^2 \tag{21}$$

to arrive at the expression

$$\tau_{\rm D} = |T_p|^2\tau_{\rm T}^{PD} + |R_p|^2\tau_{\rm R}^{QD}. \tag{22}$$

We shall examine now the real and imaginary parts of $\tau_{\rm T}^{PD}$

$$\begin{aligned}{\rm Re}\,(\tau_{\rm T}^{PD}) &= {\rm Re}\,\left([F_{\rm I}|T_p|^2]^{-1}\int_0^\infty {\rm d}p'\langle p^{(+)}|p'^{(-)}\rangle\langle p'^{(-)}|D|p^{(+)}\rangle\right)\\ &= {\rm Re}\,\left(\frac{mh}{p|T_p|^2}\int_0^\infty {\rm d}p'\, S^*_{p'p}\langle p'^{(-)}|D|p^{(+)}\rangle\right).\end{aligned} \tag{23}$$

Using (19), (6) and (11), one exactly recovers $\tau_{x{\rm T}}$, equation (12), by performing the integral in (23) with the aid of the delta functions.

Following the same steps as before with the imaginary part of $\tau_{\rm T}^{PD}$, it is found that ${\rm Im}(\tau_{\rm T}^{PD}) = -\tau_{z{\rm T}}$ so, in summary, $\tau_{\rm T}^{PD} = \tau_{x{\rm T}} - {\rm i}\tau_{z{\rm T}}$, for the transmission, and similarly, for the reflection, $\tau_{\rm R}^{QD} = \tau_{x{\rm R}} - {\rm i}\tau_{z{\rm R}}$. These complex times are equal to the ones considered by Leavens and Aers [7], or by Sokolovski and Baskin [13], and their moduli are the Büttiker–Landauer (real) times of [1, 3].

Notice that the real parts of the matrix elements of $PD = PDP + PDQ$ or $QD = QDP + QDQ$ implied by the decomposition (17) are sums of a (partial) dwell time and an interference contribution. From this perspective, since the sign of the interference contribution can be negative, negative partial times are possible [6].

Next the relation to the phase delay times is discussed. In the following we consider a real perturbation $\Delta V = \Delta V_{\rm Re}$. As stated by Huang and Wang [6], if ΔV covers the whole space, then

$$\partial\phi_{\rm T}(p)/\partial E = -\partial\phi_{\rm T}/\partial\Delta V|_{\Delta V=0} \qquad (U(x) = \Delta V\ \forall x). \tag{24}$$

Since the phase delay time $\hbar\,\partial\phi_{\rm T}(p)/\partial E$ is given formally by the same expression as $\tau_{x{\rm T}}$, equation (12), but with opposite sign, and derivatives with respect to E instead of ΔV, the above equality is proved by showing that $\partial T_p/\partial E = -\partial T_p/\partial\Delta V|_{\Delta V=0}$, and this is indeed the case, since $p = \sqrt{2m(E-\Delta V)}$. In physical terms, raising the level of the potential uniformly through the whole space keeping a constant energy for the particle is completely equivalent to lowering the energy of the particle keeping the same potential. The net effect of these two operations is to decrease the kinetic energy of the incident particle by the same amount, and the transmission coefficient can only depend on this, since the potential barrier profile remains the same. It is tempting to claim that (24) implies that the phase delay time is equal to the Larmor clock time $\tau_{x{\rm T}}$ for a magnetic field covering the whole space. In this regard we would like to point out a seeming paradox. In the previous discussion, $\tau_{x{\rm T}}$ was actually a function of the interval $[x_1, x_2]$, as can be explicitly seen in (23). When making this interval larger and larger, the spin rotation is expected to increase too, and eventually tends to infinity. (Formally, by performing the integral in (23), and taking $D \to 1$ we obtain S-matrix elements having *infinite* delta functions.) On the contrary, the phase delay time is in general a finite quantity. How can these two facts and (24) be reconciled? Physically, we are considering different processes. When enlarging the interval $[x_1, x_2]$ we are still within the ambit of scattering theory. The potential is restricted to a limited region of space (irrespective of its size) and vanishes in the asymptotic regions. The Larmor clock is related to the passage through this limited region. In contrast, when ΔV covers the whole space, no asymptotic regions exist with zero potential, and scattering theory results, in particular (8), cannot be applied. One can of course shift the zero of potential energy to ΔV by redefining the new asymptotic Hamiltonian as $\bar{H}_0 = H_0 + \Delta V$ and the new potential as $\bar{V} = V_0(x)$. However, within this framework it is the asymptotic Hamiltonian and not the potential that depends on the parameter ΔV with the consequence that, again, equation (8) does not hold. Mathematically this reflects a singular behaviour of the derivative $-\partial\phi_{\mathrm{T}}(p)/\partial\Delta V|_{\Delta V=0}$ when $x_1 = -\infty$ and $x_2 = \infty$. Taking the limit of these two positions to $\mp\infty$, respectively, does not give the same result as having them equal to $\mp\infty$ from the beginning. If one insists on regarding the phase delay times as particular cases of the Larmor clock times, it is at least worthwhile to recognize this singularity.

In summary, connections between previously defined tunnelling times in the stationary regime have been provided. A novel approach based on scattering projector operators has also been described. This method includes the complex, Larmor and absorption-based times as particular cases; it is easily generalizable to time-dependent wave packets, and makes no reference to non-standard interpretations of quantum mechanics. The physical content of the various partial times is given by the implied projector operators. In this work the emphasis has been on relating various defined times. In future work the full potential of the method will be exploited by examining the alternative times arising for the different decompositions (16), (17) and (18). This program has been partially carried out for wave packets in the time-dependent case [14].

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